Solution to the Riddler Classic of August 27, 2021

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This is a solution to the FiveThirtyEight Riddler Classic from August 27, 2021. The original problem statement may be found here [4].

First, let's measure length in units of "football-fields," so 1 ff = 100 yds = 300 ft. And then, let's measure time in units of "jarrisons," where 1 jar is the length of time that it takes Hames Jarrison to run 1 ff. So in ff/jar, Hames Jarrison's speed is exactly 1. We can also figure out the number of seconds that would equal 1 jarrison:

$$300 \text{ ft} = 1 \text{ ff} = 1 \text{ jar} \cdot \left(\frac{1 \text{ ff}}{\text{jar}}\right) = 1 \text{ jar} \cdot (15 \text{ mph})$$
$$= 1 \text{ jar} \cdot \left(\frac{15 \text{ miles}}{1 \text{ hour}}\right) \cdot \left(\frac{5280 \text{ ft}}{1 \text{ mile}}\right) \cdot \left(\frac{1 \text{ hour}}{3600 \text{ sec}}\right).$$

Solving, we find that

$$1 \text{ jar} = \frac{150}{11} \text{ sec},$$

or about 13.636 seconds. So that's how long it takes HJ to run 1 ff.

OK. Now let's set up some coordinate axes. We'll say that HJ starts at the point $(0, 0)$, so that is his position at time $t = 0$, and (measuring in ff) his target point on the opposite goal line is $(0, 1)$. So he is running in the positive y-direction. Meanwhile, the Chaser starts at the point $(1/2, 0)$.

Why did we choose to set up the axes this way? Mainly to match the conventions at this MathWorld page [3], which we refer to extensively. (By the way, that page cites work by several authors, dating back to 1732.)

Finally, we measure time starting at time $t = 0$, when HJ starts running. If he's not tackled, he'll score at time $t = 1$ (because we're measuring time in jar, and it takes him 1 jar to reach the end zone). So all the action happens between $t = 0$ and $t = 1$. Now, for $0 \le t \le 1$, Hames Jarrison's position at time t is given by

$$\vec{h}(t) = \langle 0, t \rangle.$$

For example, after half a jarrison has elapsed, he's reached midfield (and he's still running along that same sideline).

What about the Chaser? Let $\vec{r}(t) = \langle x(t), y(t) \rangle$ be the Chaser's position at time t. We know that

$$\vec{r}(0) = \left\langle \frac{1}{2}, 0 \right\rangle,$$

and also that the Chaser runs at a constant speed, say c. So

$$|\vec{r}\,'(t)| = c.$$

If the Chaser ran in a straight line, toward HJ's target point $(0,1)$, they would run diagonally from one corner of the field to the other, covering a distance of $\sqrt{1^2 + (1/2)^2} = \frac{1}{2}\sqrt{5}$ football-fields. To catch HJ, they'd have to do it in a time of 1 jarrison, so they'd have to run at a speed of at least $\frac{1}{2}\sqrt{5}$ ff/jar.

Indeed, no matter what path they take, or at what point $(0,b)$ on the sideline they manage to catch HJ, they would be better off to run straight to $(0,b)$; that way they'd be taking a shorter route, so they could afford to go slower and still reach $(0,b)$ at the same time. And, if they did take a straight-line path to $(0,b)$, if $b < 1$ then they'd be better off taking a straight-line route to $(0,1)$ instead; that way they could go even slower and still catch HJ before he scores. To travel a distance of $\sqrt{b^2 + (1/2)^2}$ football-fields, in b jarrisons, they'd need to run at a speed of

$$\frac{\sqrt{b^2 + \frac{1}{4}}}{b}$$

football-fields per jarrison, and for $b > 0$ this quantity decreases as b gets larger.

So, no matter what path the Chaser takes, if they're going to catch HJ then they must have

$$c \geq \frac{\sqrt{5}}{2} \approx 1.118.$$

In particular, we may as well assume from now on that $c > 1$; otherwise we know there is no solution. This means the Chaser has to be faster than HJ. Makes sense!

But yeah, the Chaser doesn't run in a straight line, they always run directly toward HJ. So for example at $t = 0$, the Chaser will be at $(1/2, 0)$, running in the direction of $(0,0)$, which means they're running in the negative x-direction, perpendicular to the direction HJ is running. In general,

$$\frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = \frac{\vec{h}(t) - \vec{r}(t)}{\left|\vec{h}(t) - \vec{r}(t)\right|}. \tag{1}$$

(We use unit vectors when we want to focus on direction, ignoring magnitude.)

OK, now let's follow the MathWorld link from above for a while (like, the next several pages). They note that in Equation 1, we have two ways of writing the same unit vector, and of course the dot product of a unit vector with itself is always 1. So (recalling that $|\vec{r}\,'(t)| = c$) we have

$$1 = \frac{\vec{r}\,'(t)}{c} \cdot \frac{\vec{h}(t) - \vec{r}(t)}{\left|\vec{h}(t) - \vec{r}(t)\right|}.$$

Plugging in $\vec{h}(t) = \langle 0, t\rangle$ and $\vec{r}(t) = \langle x(t), y(t)\rangle$,

$$1 = \left(\frac{1}{c}\right)\left(\frac{1}{\sqrt{x^2 + (t-y)^2}}\right)\langle x'(t), y'(t)\rangle \cdot \langle 0 - x(t), t - y(t)\rangle.$$

Let's follow MathWorld's convention of writing $\dot{x}$ to mean $\frac{dx}{dt}$, and $\dot{y} = \frac{dy}{dt}$, *etc.* (Later on we will be taking derivatives with respect to variables other than t, so we need to be somewhat careful about this.) Clearing denominators in the above equation yields

$$c\sqrt{x^2 + (t-y)^2} = -x\dot{x} + (t-y)\dot{y}.$$

Squaring both sides,

$$c^2\left(x^2 + (t-y)^2\right) = x^2\dot{x}^2 - 2x(t-y)\dot{x}\dot{y} + (t-y)^2\dot{y}^2$$
$$0 = x^2\left(\dot{x}^2 - c^2\right) - 2x(t-y)\dot{x}\dot{y} + (t-y)^2\left(\dot{y}^2 - c^2\right). \qquad (2)$$

Now recall that

$$\dot{x}^2 + \dot{y}^2 = c^2,$$

because c is the Chaser's speed, which is the magnitude of the velocity vector $\langle \dot{x}, \dot{y}\rangle$. Substituting into Equation 2 yields

$$0 = x^2\left(-\dot{y}^2\right) - 2x(t-y)\dot{x}\dot{y} + (t-y)^2\left(-\dot{x}^2\right)$$
$$0 = x^2\dot{y}^2 + 2x(t-y)\dot{x}\dot{y} + (t-y)^2\dot{x}^2$$
$$0 = \left(x\dot{y} + (t-y)\dot{x}\right)^2$$
$$0 = x\dot{y} + (t-y)\dot{x}. \qquad (3)$$

Now, if the Chaser is going to actually catch HJ, at or before $t = 1$, then (at or before $t = 1$) they need to at least make it to the line $x = 0$, since that is where HJ always is. And, once they are on that line, they will never leave it (since they will continue to move directly toward HJ, who never leaves that line). So, let T be the first time that the Chaser is on the line $x = 0$. Then for $0 < t < T$, we have $x(t) > 0$ (by the Intermediate Value Theorem). And, when $x > 0$, we have $\dot{x} < 0$.

Notice that the Chaser cannot catch HJ until at least time T.

Conceivably, they might not even catch him *at* time T; maybe they're still behind at that point—who knows? But if, at time T, the Chaser has not yet caught HJ, that'll be pretty easy for us to handle: from time T onwards, the chase is very simple (both runners moving in the same direction along a straight line, with fixed speeds 1 and c). The hard part is figuring out what happens during the time interval $0 < t < T$. So for now let's focus on that period.

For $0 < t < T$ we know $\dot{x} < 0$; therefore we can divide by it, in Equation 3:

$$0 = x\left(\frac{dy/dt}{dx/dt}\right) + (t - y)$$

$$0 = x\left(\frac{dy}{dx}\right) - y + t, \tag{4}$$

since $\frac{dy/dt}{dx/dt} = \frac{dy}{dx}$.

We now have an equation which is written almost entirely in terms of x and y; there's just a single pesky t in there. However, recall that the Chaser runs at constant speed c, hence the amount of time that has elapsed is proportional to the distance the Chaser has traveled so far. This will let us get rid of that final t.

We said that $\dot{x} < 0$ for $0 < t < T$, which means that x is a strictly decreasing function of t; hence it is one-to-one. More precisely, the map

$$t \mapsto x(t)$$

is a one-to-one function $[0, T] \to [0, 1/2]$. And also, it is onto (by the Intermediate Value Theorem). Thus there is a bijection (one-to-one correspondence) between t-values in $[0, T]$ and x-values in $[0, 1/2]$. So, since y is a function of t, it follows that y is a function of x—which simply means that the graph of the parametric curve $\langle x(t), y(t)\rangle$, restricted to t-values between 0 and T, satisfies the Vertical Line Test. For an alternative proof of this fact, suppose that the Chaser has the same x-value at two different y-values along their path. Since the Chaser can't be in two places at once, the different y-values mean that the Chaser has the same x-value at two different times, say $t_1 < t_2$. By Rolle's Theorem, there exists $t_3 \in (t_1, t_2)$ such that $dx/dt = 0$ at $t = t_3$. But that means that, at that instant t_3, the Chaser has the same x-value as HJ; so the Chaser must be on the line $x = 0$. Therefore $t_3 \geq T$. So while it may be possible for the Chaser to have the same x-value at two different times, at least one of them (t_2) must be greater than T. Thus the Chaser never has the same x-value at two different times in $[0, T]$.

In other words, if we draw the graph of the Chaser's path for $0 \leq t \leq T$ (thus excluding the part, if any, where the Chaser runs along the y-axis), that graph will satisfy the VLT, hence y is a function of x. Say, $y = f(x)$, for $0 \leq x \leq 1/2$. Further, every x-value between 0 and 1/2 does appear at some point on that graph, by the Intermediate Value Theorem, so $f(x)$ is defined for all $x \in [0, 1/2]$.

Now consider a particular instant, t_1, and suppose $0 < t_1 < T$. The distance traveled by the Chaser, from $t = 0$ to $t = t_1$, is just $c \cdot t_1$. Now let x_1 be the Chaser's x-value at that instant, so $0 < x_1 < 1/2$. Between $t = 0$ and $t = t_1$, the Chaser's x-value has decreased from 1/2 to x_1; and for each x between x_1 and 1/2, the Chaser's y-value has been just $f(x)$. Which means that the Chaser's distance traveled, from $t = 0$ to $t = t_1$, is exactly the arclength of the graph $y = f(x)$, from $x = x_1$ to $x = 1/2$. This is:

$$\int_{x_1}^{1/2} \sqrt{1 + f'(x)^2}\,dx.$$

Thus

$$c \cdot t_1 = \int_{x_1}^{1/2} \sqrt{1 + f'(x)^2}\,dx$$

$$t_1 = \frac{1}{c} \int_{x_1}^{1/2} \sqrt{1 + f'(x)^2}\, dx.$$

(Recall that we assumed $c > 1$; hence we can divide by c.) Now for clarity, let's use a different variable of integration; this doesn't affect the value of the integral:

$$t_1 = \frac{1}{c} \int_{x_1}^{1/2} \sqrt{1 + f'(\alpha)^2}\, d\alpha.$$

Now, what if we chose a different value in place of t_1, say t_2 instead? Simple, just let x_2 be the value of x that corresponds to time t_2, and then we've got

$$t_2 = \frac{1}{c} \int_{x_2}^{1/2} \sqrt{1 + f'(\alpha)^2}\, d\alpha.$$

In general, if $0 < t < T$, and x is the Chaser's horizontal position at time t, then

$$t = \frac{1}{c} \int_{x}^{1/2} \sqrt{1 + f'(\alpha)^2}\, d\alpha.$$

And of course we can switch the endpoints if we want, at the cost of a minus sign:

$$t = -\frac{1}{c} \int_{1/2}^{x} \sqrt{1 + f'(\alpha)^2}\, d\alpha. \tag{5}$$

Now recall Equation 4:

$$0 = x\left(\frac{dy}{dx}\right) - y + t.$$

In this equation also, x and y refer to $x(t)$ and $y(t)$; they refer to the Chaser's horizontal and vertical position at time t, just like in Equation 5. So, t, and x, and y, all mean exactly the same things in Equation 4 as they do in Equation 5. Therefore we can substitute, and eliminate t.

$$0 = x\left(\frac{dy}{dx}\right) - y - \frac{1}{c} \int_{1/2}^{x} \sqrt{1 + f'(\alpha)^2}\, d\alpha.$$

We'll write y' to mean $\frac{dy}{dx}$, and y'' to mean $\frac{d^2y}{dx^2}$, *etc.* So we have

$$0 = xy' - y - \frac{1}{c} \int_{1/2}^{x} \sqrt{1 + f'(\alpha)^2}\, d\alpha.$$

Now, take the derivative with respect to x on both sides, remembering the Fundamental Theorem of Calculus:

$$\begin{aligned} 0 &= y' + xy'' - y' - \frac{1}{c}\sqrt{1 + f'(x)^2} \\ &= xy'' - \frac{1}{c}\sqrt{1 + f'(x)^2}. \end{aligned}$$

(How do we know we can do this? Are we making a mistake by assuming that y is twice-differentiable? Generally d^2y/dx^2 can be written in terms of $\dot{x}, \ddot{x}, \dot{y}, \ddot{y}$; but we are told in the problem not to worry about acceleration, so are we even sure that $\ddot{x}$ and $\ddot{y}$ exist? And if not, are we potentially throwing away a valid solution by tacitly assuming that y'' exists? Actually I think this is OK: rewrite the equation as $y + \frac{1}{c}\int_{1/2}^{x} \sqrt{1 + f'(\alpha)^2}\, d\alpha = xy'$. We know that the left-hand side is differentiable, because $y' = \frac{dy/dt}{dx/dt}$ exists—recall that $\dot{x} \neq 0$—and because of the Fundamental Theorem of Calculus. So the right-hand side, being equal to the left-hand side, must be differentiable as well. But if (xy') is differentiable, then so is y', by the Quotient Rule—recalling again that $x > 0$.)

But of course, $f(x)$ was just y, written as a function of x, so $f'(x) = y'$:

$$0 = xy'' - \frac{1}{c}\sqrt{1 + (y')^2}.$$

This is a second-order differential equation, since it has a y'', and it looks pretty complicated because it has a y' inside of a square root.

But luckily, it's really just a first-order equation in disguise! Let $v = y'$. Then

$$0 = xv' - \frac{1}{c}\sqrt{1 + v^2}$$

$$\frac{1}{c}\sqrt{1 + v^2} = x\left(\frac{dv}{dx}\right).$$

This is a separable equation; see [1]. The quick-and-dirty method is to get all instances of v and dv on one side, and all instances of x and dx on the other side. Then integrate both sides:

$$\frac{1}{c}\int \frac{dx}{x} = \int \frac{dv}{\sqrt{1 + v^2}}$$

$$\frac{1}{c}\ln|x| = \ln\left|v + \sqrt{1 + v^2}\right| + K,$$

for some constant K. Now, we know that $x > 0$, since we are working over the interval $0 < t < T$. So $|x| = x$. Meanwhile, whether v is positive, negative, or zero, $(v + \sqrt{1 + v^2})$ is certainly positive. So

$$\frac{1}{c}\ln x = \ln\left(v + \sqrt{1 + v^2}\right) + K.$$

Now exponentiate both sides, so

$$x^{1/c} = e^K\left(v + \sqrt{1 + v^2}\right),$$

and let $A = e^{-K}$, so that

$$v + \sqrt{1 + v^2} = Ax^{1/c}.$$

Now

$$
\begin{aligned}
\sqrt{1+v^2} &= Ax^{1/c} - v \\
1+v^2 &= A^2x^{2/c} - 2Ax^{1/c}v + v^2 \\
1 &= A^2x^{2/c} - 2Ax^{1/c}v \\
v &= \frac{A^2x^{2/c}-1}{2Ax^{1/c}} \\
y' = v &= \frac{1}{2}\left(Ax^{1/c} - \frac{1}{Ax^{1/c}}\right) \qquad (6) \\
y &= \frac{1}{2}\int \left(Ax^{1/c} - \frac{1}{A}x^{-1/c}\right) dx \\
y &= \frac{1}{2}\left(\left(\frac{A}{1+1/c}\right)x^{1+1/c} - \left(\frac{1}{A}\right)\left(\frac{1}{1-1/c}\right)x^{1-1/c}\right) + B \\
y &= \frac{1}{2}\left(\left(\frac{Ac}{c+1}\right)x^{1+1/c} - \left(\frac{c}{A(c-1)}\right)x^{1-1/c}\right) + B. \qquad (7)
\end{aligned}
$$

Note: Our solution differs at this point from the MathWorld page, because they have assumed that $c = 1$. When $c = 1$, integrating in Equation 6 gives a qualitatively different result, involving the natural log function.

Now we need to calculate A and B, using the initial values for our problem. Of course, "initial" means at time $t = 0$, but we have eliminated t, so we need to give the initial values in terms of x instead. Well, at our initial point (starting point), the Chaser has position $(1/2, 0)$, so

$$y\big|_{x=\frac{1}{2}} = 0. \qquad (8)$$

And, at that initial instant, the Chaser is moving in the negative x-direction. That is, the tangent line to the graph $y = f(x)$, at $x = 1/2$, is horizontal, so

$$y'\big|_{x=\frac{1}{2}} = 0. \qquad (9)$$

From Equations 6 and 9, we have

$$0 = \frac{1}{2}\left(A\left(\frac{1}{2}\right)^{1/c} - \frac{1}{A\left(\frac{1}{2}\right)^{1/c}}\right).$$

Write $z = A\left(\frac{1}{2}\right)^{1/c}$, so

$$0 = z - \frac{1}{z}.$$

Thus $z^2 - 1 = 0$, so $z = \pm 1$. Therefore $A = \pm 2^{1/c}$. But, we already know that $A = e^{-K} > 0$, so $A = +2^{1/c}$.

Finally we combine this result with Equations 7 and 8, and solve for B:

$$0 = \frac{1}{2}\left(\left(\frac{2^{1/c}c}{c+1}\right)\left(\frac{1}{2}\right)^{1+1/c} - \left(\frac{c}{2^{1/c}(c-1)}\right)\left(\frac{1}{2}\right)^{1-1/c}\right) + B$$

$$0 = \left(\frac{c}{c+1}\right)\left(\frac{1}{2}\right)^1 - \left(\frac{c}{c-1}\right)\left(\frac{1}{2}\right)^1 + 2B$$

$$\frac{c}{c-1} - \frac{c}{c+1} = 4B$$

$$B = \frac{1}{2}\left(\frac{c}{c^2-1}\right).$$

So,

$$y = \frac{1}{2}\left(\left(\frac{2^{1/c}c}{c+1}\right)x^{1+1/c} - \left(\frac{c}{2^{1/c}(c-1)}\right)x^{1-1/c}\right) + \frac{1}{2}\left(\frac{c}{c^2-1}\right).$$

Now, let y_0 be the y-intercept of the graph $y = f(x)$, so $y_0 = f(0)$. That is, y_0 is the Chaser's y-value, at the time $t = T$. Thus

$$y_0 = \frac{1}{2}\left(\left(\frac{2^{1/c}c}{c+1}\right)(0)^{1+1/c} - \left(\frac{c}{2^{1/c}(c-1)}\right)(0)^{1-1/c}\right) + \frac{1}{2}\left(\frac{c}{c^2-1}\right)$$

$$y_0 = \frac{1}{2}\left(\frac{c}{c^2-1}\right). \tag{10}$$

We claim that, for $c > 1$, as c increases, y_0 decreases. This would mean that, the faster the Chaser runs, the less yardage HJ can gain before the Chaser catches him—or rather, before the Chaser reaches HJ's sideline. To show this, show that $d(y_0)/dc < 0$ for $c > 1$:

$$\frac{d(y_0)}{dc} = \left(\frac{1}{2}\right)\frac{(c^2-1)(1) - c(2c)}{(c^2-1)^2} = \frac{-c^2-1}{2(c^2-1)^2} < 0.$$

If $y_0 > 1$ then clearly the Chaser doesn't catch HJ until after he has reached the end zone, so the Chaser definitely needs $y_0 \leq 1$.

Let's solve for c in terms of y_0:

$$2y_0(c^2-1) = c$$

$$2y_0c^2 - c - 2y_0 = 0$$

$$c = \frac{1 \pm \sqrt{1+16y_0^2}}{4y_0}.$$

But, since $c > 1$, we know from Equation 10 that $y_0 > 0$. And therefore, since $c > 0$, we conclude that

$$c = \frac{1 + \sqrt{1+16y_0^2}}{4y_0}.$$

We want c to be as small as possible, while still catching HJ. But smaller c-values correspond to larger y_0-values. The largest that y_0 can possibly be, while catching HJ, is 1, so the smallest possible c is

$$c = \frac{1+\sqrt{17}}{4} \approx 1.280776.$$

As a check, we said before that c must be at least $\sqrt{5}/2$. But, notice that

$$\frac{1+\sqrt{17}}{4} > \frac{1+4}{4} = \left(\frac{\sqrt{5}}{2}\right)^2,$$

and $(\sqrt{5}/2)^2 > \sqrt{5}/2$ since $\sqrt{5}/2 > \sqrt{4}/2 = 1$. (If $x > 1$ then it follows that $x \cdot x > x \cdot 1$, since $x > 0$.)

So, we know that we need $c \geq (1+\sqrt{17})/4$; any slower and we get $y_0 > 1$. But, does this speed actually work to catch HJ, or is it still not enough?

The key is that we now see that there is no "second part" of the Chaser's path; the Chaser meets HJ at the same time that they reach the line $x = 0$: at time $t = T = 1$. Indeed, recall Equation 4, which said:

$$t = y - x\left(\frac{dy}{dx}\right).$$

Plugging in our results for y and y', we have

$$t = \frac{1}{2}\left(\left(\frac{2^{1/c}c}{c+1}\right)x^{1+1/c} - \left(\frac{c}{2^{1/c}(c-1)}\right)x^{1-1/c} + \left(\frac{c}{c^2-1}\right)\right) - \frac{1}{2}\left(2^{1/c}x^{1+1/c} - \frac{x^{1-1/c}}{2^{1/c}}\right). \tag{11}$$

In particular, at $x = 0$ (or, as $x \to 0^+$) we have

$$t = \frac{c}{2(c^2-1)},$$

and setting $c = (1+\sqrt{17})/4$ yields $t = 1$. A very similar check shows that at $x = 0$, we get $y = 1$. Therefore the Chaser will meet HJ, at the point $(0, 1)$, at time $t = 1$—so this value of c is *just* sufficient to meet HJ before (or, at the instant that) he reaches the end zone.

So, our answer is that the Chaser must be faster than HJ, by a factor of at least $(1+\sqrt{17})/4$. That is, their speed must be at least

$$\left(\frac{1+\sqrt{17}}{4}\right) \cdot (15 \text{ mph}),$$

or about 19.212 miles per hour.

Another way to reach Equation 11 is to use the information from the initial problem statement: suppose that at time t_1, the Chaser is at the point (x_1, y_1); meanwhile HJ is at $(0, t_1)$. Thus the line tangent to the curve $y = f(x)$, at (x_1, y_1), must go through $(0, t_1)$. Now,

$$y_1 = \frac{1}{2}\left(\left(\frac{2^{1/c}c}{c+1}\right)x_1^{1+1/c} - \left(\frac{c}{2^{1/c}(c-1)}\right)x_1^{1-1/c} + \frac{c}{c^2-1}\right),$$

and meanwhile the slope of the line is the value of y' at x_1, namely

$$\frac{1}{2}\left(2^{1/c}x_1^{1/c} - 2^{-1/c}x_1^{-1/c}\right) = \frac{1}{2}\left((2x_1)^{1/c} - (2x_1)^{-1/c}\right).$$

Thus the equation of the tangent line is

$$y - \frac{1}{2}\left(\left(\frac{2^{1/c}c}{c+1}\right)x_1^{1+1/c} - \left(\frac{c}{2^{1/c}(c-1)}\right)x_1^{1-1/c} + \frac{c}{c^2-1}\right) = \frac{1}{2}\left((2x_1)^{1/c} - (2x_1)^{-1/c}\right)(x - x_1).$$

Since the line must go through $(0, t_1)$, we have

$$t_1 = \frac{1}{2}\left(\left(\frac{2^{1/c}c}{c+1}\right)x_1^{1+1/c} - \left(\frac{c}{2^{1/c}(c-1)}\right)x_1^{1-1/c} + \frac{c}{c^2-1}\right) + \frac{1}{2}\left((2x_1)^{1/c} - (2x_1)^{-1/c}\right)(-x_1).$$

This matches Equation 11.

We can simplify this equation a bit; on the right-hand side, multiply and divide by 2. Then

$$t_1 = \frac{1}{4}\left(\left(\frac{c}{c+1}\right)(2x_1)^{1+1/c} - \left(\frac{c}{c-1}\right)(2x_1)^{1-1/c} + \frac{2c}{c^2-1} - (2x_1)^{1+1/c} + (2x_1)^{1-1/c}\right)$$

$$t_1 = \frac{1}{4}\left(\left(\frac{c}{c+1} - 1\right)(2x_1)^{1+1/c} - \left(\frac{c}{c-1} - 1\right)(2x_1)^{1-1/c} + \frac{2c}{c^2-1}\right).$$

Now recall that $c = (1+\sqrt{17})/4$ yields $c/(2(c^2-1)) = 1$, hence (since x_1 was arbitrary)

$$t = 1 - \frac{1}{4}\left(\left(\frac{1}{c+1}\right)(2x)^{1+1/c} + \left(\frac{1}{c-1}\right)(2x)^{1-1/c}\right).$$

At $x = 0$ we have $t = 1$, and at $x = 1/2$ we have $t = 1 - 2c/(4(c^2-1) = 0$, as expected. Also, with $c = (1+\sqrt{17})/4$, we find that

$$\frac{1}{c+1} = \frac{5-\sqrt{17}}{2} = 2\left(1 - \frac{1}{c}\right),$$

and

$$\frac{1}{c-1} = \frac{3+\sqrt{17}}{2} = 2\left(1 + \frac{1}{c}\right).$$

Thus, if we write $\alpha = 1 + 1/c$, and $\beta = 1 - 1/c$, then

$$t = 1 - \frac{1}{2}\left(\beta(2x)^{\alpha} + \alpha(2x)^{\beta}\right).$$

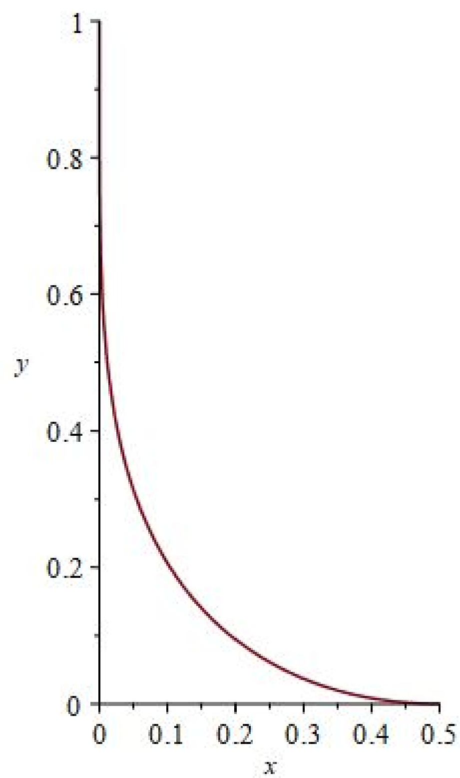
Thus, for any given $t \in (0, 1)$, we could probably use Newton's Method (with a suitably close initial guess), or a binary search, or a similar algorithm to determine the Chaser's position at time t to any desired number of decimal places, but this is likely the best we can do. In particular, binary search seems fairly robust for this application, while Newton's Method seems rather sensitive to the initial guess, and may fail to converge for poor guesses. Or, it may even converge to the wrong value! For example, when $x = 0.01$ we have $t = 0.6222156786834542$. However, using Newton's Method to solve for the x-value which yields $t = 0.6222156786834542$, with initial guess $x_0 = 0.25$, does not seem to converge to 0.01; instead, my computer reaches the non-real value

$$-0.8436799292068115 - 1.3007913488407918i.$$

(It first overshoots the desired 0.01, reaching a negative number, and then raises that number to a non-integer value, causing it to leave the real line.)

On the other hand, while it may take significant computation to say "where the Chaser is at time t," (*i.e.*, determine x for a given t), it is perfectly straightforward to determine "when the Chaser reaches the horizontal position x." For example, when $x = 1/4$, the Chaser is halfway between the sidelines; this happens at $t \approx 0.203232$. When $x = 0.1$ the Chaser has covered 4/5 of the horizontal distance across the field, and this happens at $t \approx 0.368086$. So the Chaser spends most of their time in close proximity to HJ's sideline.

We close with a plot of the Chaser's path, created in Maple [2].



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[2] Maple 2019.0. Maplesoft, a division of Waterloo Maple Inc., Waterloo, Ontario.

[3] Weisstein, Eric W. "Pursuit Curve." From MathWorld—A Wolfram Web Resource. https://mathworld.wolfram.com/PursuitCurve.html

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