Introduction to Point-Set Topology

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Fall 2011
Outline

Metric Spaces

Topological Spaces
  Definitions
  Continuous Functions
Given any two real numbers $x$ and $y$, we define the distance between $x$ and $y$ to be:

$$d(x, y) = |x - y|$$

Example

The distance from 2 to 6 is

$$d(2, 6) = |2 - 6| = |-4| = 4$$

This "distance" function $d$ takes a pair of real numbers $(x, y)$ and returns a single real number. Written formally:

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The distance function \( d(x, y) = |x - y| \) has several useful properties:

- \( d(x, y) \geq 0 \).
- \( d(x, y) = 0 \) if and only if \( x = y \).
- \( d(x, y) = d(y, x) \).
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Example: Suppose we need to find all numbers which lie less than distance 3 away from the number 5. Then we wish to solve:

$$|x - 5| < 3 \iff 2 < x < 8$$

The set of points whose distance from 5 is less than 3 is the open interval $(2, 8)$.
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The plane $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ also has a distance function, $D$. This function takes two points $(a, b)$ and $(x, y)$ in $\mathbb{R}^2$ and returns one nonnegative real number.
Two-dimensional real space

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$D: \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}_{\geq 0}$ is defined by:

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We describe the set of points $(x, y)$ in $\mathbb{R}^2$ which lie fewer than $r$ units away from the point $(a, b)$:
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The points $(x, y)$ which satisfy this inequality make up the open disc of radius $r$, centered at $(a, b)$. 
The function

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satisfies the same useful properties as the absolute value function \( d \) did:

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3. \( D((x, y), (a, b)) = D((a, b), (x, y)) \)
4. \( D((x, y), (a, b)) \leq D((x, y), (z, w)) + D((z, w), (a, b)) \) (Triangle Inequality)
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Definition

A metric space is a set $X$, along with a function $d : X \times X \rightarrow \mathbb{R}_{\geq 0}$, satisfying:

1. $d(x, y) = 0$ if and only if $x = y$.
2. $d(x, y) = d(y, x)$.
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The function $d$ is called a metric on $X$. The elements of $X$ are called points.

Example

We have seen that the functions $d$ and $D$ are metrics on the real line $\mathbb{R}$ and the real plane $\mathbb{R}^2$, respectively. (These metrics $d$ and $D$ are called the "Euclidean" metrics on $\mathbb{R}$ and $\mathbb{R}^2$, respectively.)
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Assuming there are no one-way roads or other oddities, any system of roads is a metric space, under the metric

\[ d(x, y) = \text{the length of the shortest route from } x \text{ to } y. \]
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Then \( d \) is a metric on \( X \), called the \textit{discrete metric}. 
Example

Let $\mathbb{F}_2^n$ be the set of all “words” of length $n$, where every “letter” must be either a “0” or a “1”. For instance, $(0, 1, 1, 1, 0, 1)$ and $(1, 1, 0, 0, 1, 1)$ are elements of $\mathbb{F}_2^6$. 

Proof: Let $x$ and $y$ be words in $\mathbb{F}_2^n$. Clearly $d(x, y)$ is a nonnegative integer; hence a nonnegative real number. Certainly $d(x, y) = d(y, x)$. If $x = y$ (that is, $x$ and $y$ are the same word), then $x$ and $y$ will not differ in any entry, in which case $d(x, y) = 0$. However, if $x \neq y$, then $x$ and $y$ will differ in at least one entry, so $d(x, y) > 0$. We have the Triangle Inequality left to prove.

Exercise. □

What if our alphabet has more than 2 letters (say, 26)?
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Let \( F^n_2 \) be the set of all “words” of length \( n \), where every “letter” must be either a “0” or a “1”. For instance, \((0, 1, 1, 1, 0, 1)\) and \((1, 1, 0, 0, 1, 1)\) are elements of \( F^6_2 \). Then \( F^n_2 \) is a metric space under the metric \( d(x, y) = \) the number of entries in which the words \( x \) and \( y \) differ.

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- $d_5((x, y), (a, b)) = \begin{cases} 0 & \text{if } (x, y) = (a, b) \\ 2 & \text{if } (x, y) \neq (a, b) \end{cases}$
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- $d_2((x, y), (a, b)) = \max(|x - a|, |y - b|)$
- $d_3((x, y), (a, b)) = |x - a| + |y - b|$
- $d_4((x, y), (a, b)) = \begin{cases} 0 & \text{if } (x, y) = (a, b) \\ \frac{1}{2} & \text{if } (x, y) \neq (a, b) \end{cases}$
- $d_5((x, y), (a, b)) = \begin{cases} 0 & \text{if } (x, y) = (a, b) \\ 2 & \text{if } (x, y) \neq (a, b) \end{cases}$
- $d_6((x, y), (a, b)) = (|x - a| + 1) \ast (|y - b| + 1) - 1$
Which of the following functions are metrics on the real plane $\mathbb{R}^2$?

- $d_1((x, y), (a, b)) = \sqrt{(x - a)^2 + (y - b)^2}$ Euclidean metric
- $d_2((x, y), (a, b)) = \max(|x - a|, |y - b|)$ chessboard metric
- $d_3((x, y), (a, b)) = |x - a| + |y - b|$ taxicab metric
- $d_4((x, y), (a, b)) = \begin{cases} 0 & \text{if } (x, y) = (a, b) \\ \frac{1}{2} & \text{if } (x, y) \neq (a, b) \end{cases}$ discrete metric
- $d_5((x, y), (a, b)) = \begin{cases} 0 & \text{if } (x, y) = (a, b) \\ 2 & \text{if } (x, y) \neq (a, b) \end{cases}$ discrete metric
- $d_6((x, y), (a, b)) = (|x - a| + 1) \times (|y - b| + 1) - 1$ Not a metric—why?
Definition

Let $X$ be a metric space, under the metric $d$, and let $x \in X$.

Thus $B_r(x)$ is the set of points which are distance less than $r$ away from the point $x$. A subset $U$ of $X$ is called open if for every $u \in U$, there exists $r > 0$ such that $B_r(u) \subseteq U$. 

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Introduction to Point-Set Topology
Definition

Let \( X \) be a metric space, under the metric \( d \), and let \( x \in X \).

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Example

Recall that $\mathbb{R}$ is a metric space under $d(x, y) = |x - y|$.
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The open ball in $\mathbb{R}$ of radius $r = 3$, centered at $x = 5$, is the open interval $(2, 8)$. 

The interval $U = (3, 4)$ is open in $\mathbb{R}$, since for any $u \in U$ there is an $r$ such that the open ball $B_r(u)$ is contained in $U$: we can take $r = \frac{1}{2} \min(|u - 3|, |u - 4|)$.

For instance, suppose $u = 3.1 \in (3, 4) = U$. Then letting $r = 0.05$, we have $B_{0.05}(3.1) = (3.05, 3.15) \subseteq (3, 4)$. 

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Exercise: For each of the metrics $d_1, \ldots, d_5$ defined in the previous exercise, what does the open ball of radius 1, centered at $(0,0)$, look like?
A word on intersections

Let $A_1, A_2$ be sets. We define

$$A_1 \cap A_2 = \{ x \mid x \in A_1 \text{ and } x \in A_2 \} = \{ x \mid x \text{ is in both } A_1 \text{ and } A_2 \}$$
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Proof of (ii) and (iii) (others left as exercises):

- ii) This statement is true vacuously; if it were false, then there would need to be an \(x \in \emptyset\) for which we could not find an appropriate \(B_{r}(x)\). No such \(x\) exists, so we are done.
- iii) Let \(x \in U_1 \cap \ldots \cap U_n\). Each \(U_i\) is open, so there are positive real numbers \(r_1, r_2, \ldots, r_n\), such that \(B_{r_i}(x) \subseteq U_i\) for each \(i\). Let \(r = \min(r_1, \ldots, r_n)\). Then \(B_{r}(x) \subseteq U_i\) for each \(i\), so \(B_{r}(x) \subseteq \bigcap_{i=1}^n U_i\). □
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A topological space consists of a set $X$, together with a set $\tau$ whose elements are subsets of $X$, such that:

- $X \in \tau$ and $\emptyset \in \tau$
- If $U_1, U_2, \ldots, U_n \in \tau$, then $\bigcap_{i=1}^{n} U_i \in \tau$. (That is, $\tau$ is closed under finite intersections.)
- If $U_i \in \tau$ for all $i \in I$, then $\bigcup_{i \in I} U_i \in \tau$. (That is, $\tau$ is closed under arbitrary unions.)

The distinction between "finite union" and "arbitrary union" is only important if $\tau$ includes infinitely many subsets of $X$, which is only possible if $X$ is an infinite set. The elements of $X$ are called points, and the elements of $\tau$ are called open sets. The set $\tau$ is called a topology on $X$. 

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Examples

- Let $X$ be any (nonempty) set, and let $\tau$ be the set of all subsets of $X$. Then $(X, \tau)$ is a topological space. (This $\tau$ is called the “discrete” topology on $X$.)
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- More generally, any metric space $X$ is a topological space, where the open sets are unions of open balls. Thus topological spaces are a generalization of metric spaces.
More examples

Let $X = \{a, b\}$, and let $\tau = \{\emptyset, \{a\}, \{a, b\}\}$. Then $(X, \tau)$ is a topological space, called the Sierpinski space.
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Let $X$ be any set, and let $\tau = \{\emptyset, X\}$. Then $(X, \tau)$ is a topological space. This $\tau$ is called the indiscrete topology on $X$. 

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- Let $\mathbb{Z}$ be the set of integers. Is the set $\tau$ of all infinite subsets of $\mathbb{Z}$ a topology on $\mathbb{Z}$?
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- Let $\mathbb{R}$ be the set of real numbers. Let $\tau$ be the set of unions of half-open intervals $[a, b)$. Is $\tau$ a topology on $\mathbb{R}$?
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Definition

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For example, consider the usual euclidean topology on $\mathbb{R}$. Is the interval $[a, b]$ a closed set? Yes: its complement is $(-\infty, a) \cup (b, \infty)$, which is an open set in this topology: it is a union of open intervals.
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Let \((X, \tau)\) be a topological space. As we said, the elements of \(\tau\) are subsets of \(X\), called open sets. A subset \(A\) of \(X\) is called closed if its complement \(X - A\) is open.

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We said that for any topological space \(X\), the sets \(\emptyset\) and \(X\) are open. But they are also closed! Sets that are both open and closed are sometimes called “clopen” sets. :) Exercise: prove that in a discrete topological space \(X\), every subset of \(X\) is clopen.
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If we had wanted, we could have defined the term “topological space” to be a set $X$ along with a set $\sigma$ of closed sets, satisfying the properties above.
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The open sets in this topological space are called cofinite, since their complements are finite. This topology is called the cofinite topology on $\mathbb{Z}$, and in fact we can define the cofinite topology on any set, not just $\mathbb{Z}$.
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These days the Zariski topology is especially important, since it is the starting point for the entire area of Algebraic Geometry, which is a very active research area right now. We won’t describe these spaces here, but if you go to graduate school in math, it is very likely that you will encounter the Zariski topology!
Definition

Let $X$ and $Y$ be topological spaces. A function $f : X \to Y$ is continuous if:

The notation here doesn’t necessarily mean that $f$ has an inverse. Here $f^{-1}(U)$ is a set:

$$f^{-1}(U) = \{ x \in X | f(x) \in U \}.$$ 

Thus $f^{-1}(U)$ is the set of points that are mapped into $U$ by $f$. 

Dan Swenson, Black Hills State University

Introduction to Point-Set Topology
Definition

Let $X$ and $Y$ be topological spaces. A function $f : X \rightarrow Y$ is **continuous** if:

- for each open subset $U$ of $Y$, the preimage $f^{-1}(U)$ is an open subset of $X$.

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Thus $f^{-1}(U)$ is the set of points that are mapped into $U$ by $f$. 
Example: Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be defined by $f(x) = x^3$. Then

$\{x \in \mathbb{R} \mid x^3 \in (-8, 8)\} = \{x \in \mathbb{R} \mid -2 < x < 2\} = (-2, 2)$. 

Exercise: Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be given by $f(x) = \sqrt{x} + 1$. Find $f^{-1}(3, \infty)$. 
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since any \( x \) in the interval \((-2, 2)\) will be sent into the interval \((-8, 8)\), and no other \( x \)-values will land in \((-8, 8)\).
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The idea is that a continuous function $f : \mathbb{R} \to \mathbb{R}$ can take open sets to open sets, like $f(x) = x^3$,
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The idea is that a continuous function $f : \mathbb{R} \to \mathbb{R}$ can take open sets to open sets, like $f(x) = x^3$, and it can take open sets to non-open sets, like $f(x) = x^2$, which takes $(-\infty, \infty)$ to $[0, \infty)$, but it will never take a non-open set to an open set, as we’ll see on the next slide.
Theorem

Let $f : \mathbb{R} \to \mathbb{R}$ be any function. Then $f$ is continuous if and only if, given any open subset $U$ of $\mathbb{R}$, the preimage $f^{-1}(U)$ is also open.
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Proof: See supplement.
**Theorem**

Let \( f : \mathbb{R} \to \mathbb{R} \) be any function. Then \( f \) is continuous if and only if, given any open subset \( U \) of \( \mathbb{R} \), the preimage \( f^{-1}(U) \) is also open.

Proof: See supplement.

This definition using open sets allows us to define continuity for functions between any two topological spaces, not just metric spaces.
Example: Let $X$ be the set $\{1, 2\}$. We can define (at least) two different topologies on this set:
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Let $\tau$ be the discrete topology, $\tau = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$, and let $\sigma$ be the indiscrete topology, $\sigma = \{\emptyset, \{1, 2\}\}$. 

Let $f : (X, \tau) \to (X, \sigma)$ be the identity function, $f(x) = x$. Is $f$ continuous?

We need to check whether the preimage of an open set in $(X, \sigma)$ is always an open set in $(X, \tau)$.

The only open sets in $(X, \sigma)$ are $\emptyset$ and $X = \{1, 2\}$.

Clearly, $f^{-1}(\emptyset) = \emptyset$, which is open.

Also, $f^{-1}(X) = X$ (exercise), so $f$ is continuous.

Now let $g : (X, \sigma) \to (X, \tau)$ be defined by $g(x) = x$. Then $g$ is not continuous.
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As you might expect, the composition $f(g(x))$ is continuous if $f$ and $g$ are both continuous.
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This will basically mean that the collection of ALL topological spaces forms a category, but we won't discuss that today.
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We will be content to define the idea of a homeomorphism between two topological spaces:

**Definition**

*Let $X$ and $Y$ be two topological spaces. We say that $X$ and $Y$ are homeomorphic if there exist continuous functions $f : X \to Y$ and $g : Y \to X$ which are inverses of each other. The functions $f$ and $g$ are called homeomorphisms.*
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In the example on the last slide, we had $f : X \to Y$ and $g : Y \to X$, and they were inverse to each other: $(f(g(x)) = x$ and $g(f(y)) = y$. 
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In the example on the last slide, we had $f : X \to Y$ and $g : Y \to X$, and they were inverse to each other: $(f(g(x))) = x$ and $g(f(y)) = y$. However, $g$ was not continuous, so these functions were not homeomorphisms.
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Proof: We need continuous functions $f : (0, 1) \to (0, 2)$ and $g : (0, 2) \to (0, 1)$ which are inverse to each other.

Let $f(x) = 2x$, and $g(y) = \frac{y}{2}$.

Show that $f$ and $g$ are send these intervals to each other, that they are inverses, and that they are both continuous (Exercise).

Comment: A homeomorphism of topological spaces is very much like an isomorphism of groups (or rings, etc.).
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