Explicit Computation of Cohomological Hodge Parameters for Calabi-Yau Threefolds in Type IIA and Type IIB String Theory

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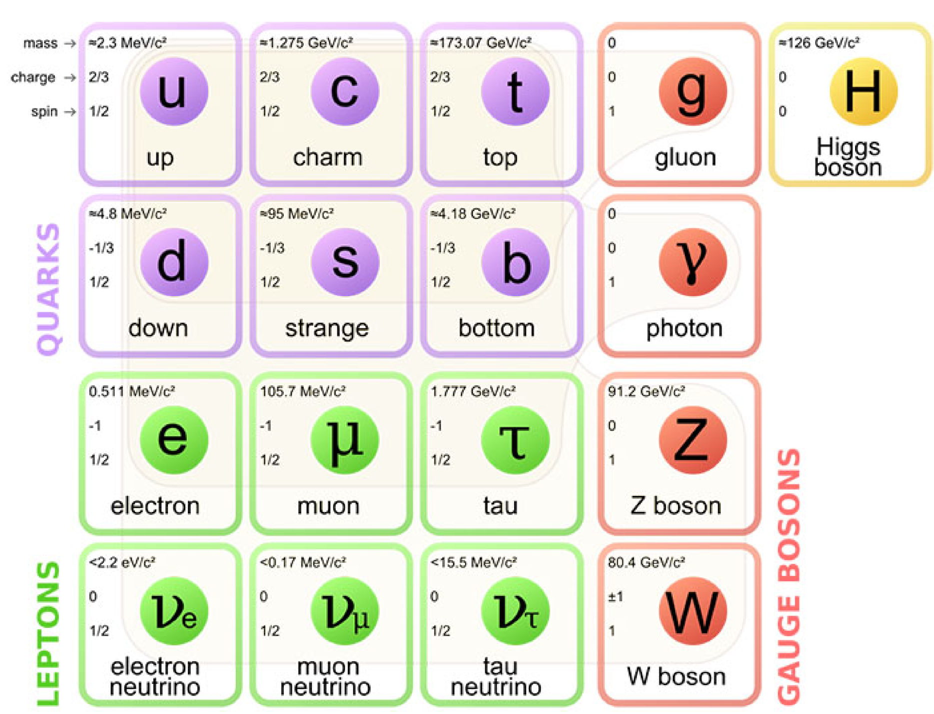
Introduction / Physics Motivation

Throughout the twentieth century, two fundamental frameworks dominated our understanding of Physics: Einstein's General Theory of Relativity [GTR] and Quantum Mechanics [QM]. From Quantum Mechanics or to be more precise Quantum Field Theory came the now famous Standard Model [SM] of Particle Physics, which describes all of the known elementary particles and three of the four known fundamental forces. The missing force, gravity, is described by General Theory of Relativity. A theory which describes all four fundamental forces, sometimes called a Grand Unified Theory + GTR, has proven to be elusive; however, String Theory has emerged as a possible solution. In Type IIA and IIB String Theory, particles are described by strings residing in 10 dimensions: the four of Minkowski space-time and six dimensions of a Calabi-Yau manifold. Through an in-depth study of the works of Witten, Candelas, Katz, Kodaira, Hirzebruch and others, we have explicitly computed the cohomological Hodge parameters used to describe two different six-dimensional Calabi-Yau manifolds producing a framework in which Einstein's Field Equations in a vacuum are true, and where three and four generations of particles are predicted, thus combining General Theory of Relativity with the Standard Model.



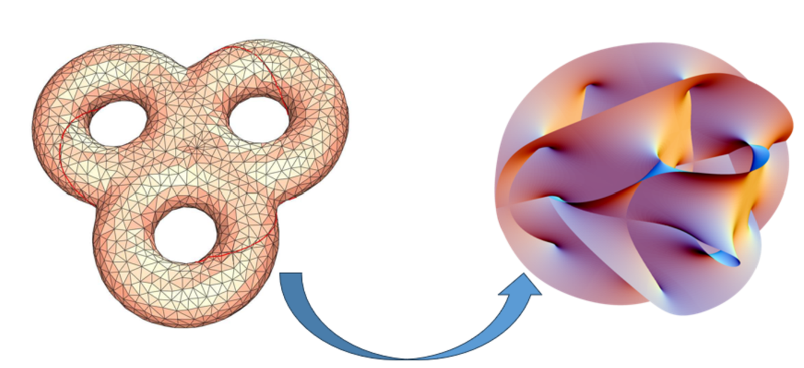
Brief Description of Type IIA-IIB String Theory

Σ is a Riemann surface generated by a closed string traveling through Minkowski Space-Time and let X be a Calabi-Yau [CY] threefold. Let $\phi^i : \Sigma \to X, i = 1,2,3$ such that $\phi^i = \phi^i(z,\overline{z})$ and $\phi^{\overline{i}} = \overline{\phi^i(z,\overline{z})}, \overline{i} = 1,2,3$ describes the bosons. By applying supersymmetry, there exists a fermion for every boson such that for ϕ^i there is ψ^i_+ and ψ^i_- where + represents the quantum spin number $+1/2$ and - represents the spin number $-1/2$. Similarly, for $\phi^{\overline{i}}$ there is $\psi^{\overline{i}}_+$ and $\psi^{\overline{i}}_-$. For the sake of this discussion, let $X = \mathbb{C}^3$ which is **a trivial CY manifold**. Then the action $S = 2t\int_\Sigma(\frac{1}{2}g_{i\overline{j}}(\frac{\partial\phi^i}{\partial z}\frac{\partial\phi^{\overline{j}}}{\partial\overline{z}} + \frac{\partial\phi^{\overline{j}}}{\partial z}\frac{\partial\phi^i}{\partial\overline{z}}) + ig_{i\overline{j}}\psi^{\overline{j}}_-\frac{\partial\psi^i_-}{\partial z} + ig_{i\overline{j}}\psi^{\overline{j}}_+\frac{\partial\psi^i_+}{\partial\overline{z}}$ where $g_{i\overline{j}}$ is the hermitian metric on X. The Supersymmetric transformations are given by $\delta\phi^i = i\alpha_-\psi^i_+ + i\alpha_+\psi^i_-$, $\delta\phi^{\overline{i}} = i\alpha'_-\psi^{\overline{i}}_+ + i\alpha'_+\psi^{\overline{i}}_-$, $\delta\psi^i_+ = -\alpha'_-\frac{\partial\phi^i}{\partial z}$, $\delta\psi^{\overline{i}}_+ = -\alpha_-\frac{\partial\phi^{\overline{i}}}{\partial z}$, $\delta\psi^i_- = -\alpha'_+\frac{\partial\phi^i}{\partial\overline{z}}$, $\delta\psi^{\overline{i}}_- = -\alpha_+\frac{\partial\phi^{\overline{i}}}{\partial\overline{z}}$. The parameters $\alpha_+, \alpha_-, \alpha'_+, \alpha'_-$ are called Grassmann or Fermionic parameters. Hence this theory is called $N = 2$ or $N = (2,2)$ string theory. A-Twist: $\alpha_- = \alpha'_+ = 0$ and $\alpha_+ = \alpha'_- = \alpha$. Thus, the A-twist results in $\psi^i_+ : \Sigma \to K_\Sigma^{1/2} \otimes \phi^*(T_{(1,0)}X)$ becoming $\chi^i : \Sigma \to \phi^*(T_{(1,0)}X)$, $\psi^{\overline{i}}_+ : \Sigma \to K_\Sigma^{1/2} \otimes \phi^*(T_{(0,1)}X)$ becoming $\psi^{\overline{i}}_z : \Sigma \to K_\Sigma \otimes \phi^*(T_{(0,1)}X)$, $\psi^i_- : \Sigma \to \overline{K}_\Sigma^{1/2} \otimes \phi^*(T_{(1,0)}X)$ becoming $\psi^i_{\overline{z}} : \Sigma \to \overline{K}_\Sigma \otimes \phi^*(T_{(1,0)}X)$, $\psi^{\overline{i}}_- : \Sigma \to \overline{K}_\Sigma^{1/2} \otimes \phi^*(T_{(0,1)}X)$ becoming $\psi^{\overline{i}} : \Sigma \to \phi^*(T_{(0,1)}X)$ B-Twist: $\alpha_+ = \alpha_- = 0$ and $\alpha'_+ = \alpha'_- = \alpha$. Thus, the B-twist results in ψ^i_+ becoming ψ^i_z, ψ^i_- becoming $\psi^i_{\overline{z}}$, $\psi^{\overline{i}}_+$ becoming $\psi^{\overline{i}}_1$, $\psi^{\overline{i}}_-$ becoming $\psi^{\overline{i}}_2$. where K_Σ is the canonical line bundle and $K_\Sigma^{1/2} \otimes K_\Sigma^{1/2} \cong K_\Sigma$.

Observables of Type IIA model on X are equivalent to observables of Type IIB model on X^* where X^* is the Mirror manifold.

Mathematical Preliminaries

$\mathbb{CP}^n$ is an n-dimensional complex projective space defined as $\mathbb{CP}^n = \{z = (z_0 : z_1 : ... : z_n) \in \mathbb{C}^{n+1} | z \neq \overline{0}\}/\sim$ where $\overline{0} = (0,0,...0)$. The equivalence relation, $\sim$, is defined as $z \sim z'$ iff $\forall\lambda \in \mathbb{C}\backslash\{0\}$, $z' = \lambda z$. $\mathbb{CP}^n$ can be described as a union of open sets $U_i = \{(z_0,...,z_i,...,z_n) \in \mathbb{CP}^n | z_i \neq 0\}\ \forall\ 0 \leq i \leq n$. Let $\phi_i : U_i \to \mathbb{C}^n$ defined by $\phi_i(z_0,...,z_i,...,z_n) = (\frac{z_o}{z_i}, \frac{z_1}{z_i}, ..., \hat{1}, ...\frac{z_n}{z_i})$. Then, (U_i, ϕ_i) is a chart and $\{(U_i,\phi_i) : i = 0,1,...,n\}$ is an atlas. $X = \{(z_0 : z_1 : ... : z_n) \in \mathbb{CP}^n \mid F(z_0, z_1, ..., z_n) = 0\}$ is a sub-manifold of $\mathbb{CP}^n$ defined by the homogeneous polynomial F of degree d. $dim(X) = n - 1$. X is a compact sub-manifold of $\mathbb{CP}^n$. Every compact sub-manifold of $\mathbb{CP}^n$ is an algebraic manifold. T_zX is the tangent space to X where $B_{T_zX} = \left\{\frac{\partial}{\partial z_0}, \frac{\partial}{\partial z_1}, ..., \frac{\partial}{\partial z_n}, \frac{\partial}{\partial\overline{z_0}}, \frac{\partial}{\partial\overline{z_1}}, ..., \frac{\partial}{\partial\overline{z_n}}\right\}$. T^*_zX is the dual space of the tangent space where $B_{T^*_zX} = \{dz_0, dz_1, ..., dz_n, d\overline{z_0}, d\overline{z_1}, ..., d\overline{z_n}\}$. $B_{\bigwedge^n T^*_{z,(1,0)}X} = \{dz_0 \wedge ... \wedge dz_n\}$, where $\bigwedge^n T^*_{z,(1,0)}X$ is the n^{th} exterior product vector space. Let $\Omega^{(p,q)}(X)$ be a complex vector space of differential form of bi-degree (p,q) such that locally, $\forall\ \omega \in \Omega^{(p,q)}(X)$, $\omega = f^{i_1,...,i_p\overline{i_1}...\overline{j_q}}dz_{i_1} \wedge \cdots dz_{i_p} \wedge d\overline{z}_{\overline{i_1}} \wedge \cdots d\overline{z}_{\overline{i_q}}$ with exterior derivative $d = dz^i \wedge \partial_i + d\overline{z}_{\overline{i}} \wedge \partial_{\overline{i}}$. A metric can be defined as $g : T^*X \times T^*X \to \mathbb{R}$, where $g = g^{\overline{i}j}d\overline{z}_{\overline{i}} \otimes z_j + g^{i\overline{j}}dz_i \otimes d\overline{z}_{\overline{j}}$. By quotienting, we obtain $\omega = 2ig^{i\overline{j}}dz_i \wedge d\overline{z}_{\overline{j}} \in \Omega^{(1,1)}(X)$. If $d\omega = 0$, then X is called a Kähler manifold and ω is called a Kähler form and is said to be closed. Furthermore, $d^2 = 0$, $\partial^2 = 0$, and $\overline{\partial}^2 = 0$. Also, note that $\mathbb{CP}^n$ is Kähler. By Kodaira, any sub-manifold of a Kähler manifold is also Kähler. $\Omega^{(p,q)}(X) \xrightarrow{\overline{\partial}} \Omega^{(p,q+1)}(X)$. Similarly, $\Omega^{(p,q)}(X) \xrightarrow{\partial} \Omega^{(p+1,q)}(X)$. The cochain complex of the space of (p,q)-forms is $\Omega^{(p,0)}(X) \xrightarrow{\overline{\partial}} \Omega^{(p,1)}(X) \xrightarrow{\overline{\partial}} ... \xrightarrow{\overline{\partial}} \Omega^{(p,q)}(X) \xrightarrow{\overline{\partial}} ... \xrightarrow{\overline{\partial}} \Omega^{(p,n)}(X) \xrightarrow{\overline{\partial}} 0$. The Dolbeault Cohomology, $H^{p,q}_{\overline{\partial}}(X)$, is defined to be the following quotient space: $H^{p,q}_{\overline{\partial}}(X) = ker(\overline{\partial} : \Omega^{(p,q)}(X) \to \Omega^{(p,q+1)}(X))/im(\overline{\partial} : \Omega^{(p,q-1)}(X) \to \Omega^{(p,q)}(X))$. A Hodge parameter is defined as $h^{p,q}(X) = dim_{\mathbb{C}}(H^{p,q}_{\overline{\partial}}(X))$



$h^{0,0}$

$h^{1,0}$ $h^{0,1}$

$h^{2,0}$ $h^{1,1}$ $h^{0,2}$

$h^{3,0}$ $h^{2,1}$ $h^{1,2}$ $h^{0,3}$

$h^{3,1}$ $h^{2,2}$ $h^{1,3}$

$h^{3,2}$ $h^{2,3}$

$h^{3,3}$

Calabi-Yau Manifold

Definition 1: *A complex n-dimensional compact Kähler manifold X whose canonical bundle $\bigwedge^n T^*_{(1,0)}X$ has a nowhere vanishing holomorphic section, that is, $s : X \to \bigwedge^n T^*_{(1,0)}X$ locally given by $s = f(z_1, \ldots, z_n)dz_1 \wedge \cdots \wedge dz_n$ is non-vanishing section is called $Calabi - Yau$ manifold*

Remark: *The first Chern class defined by $c_1(X) = c_1(\bigwedge^n T^*_{(1,0)}X) = [Z(s)]$ where $Z(s) = \{z \in X : s(z) = 0\}$ where s is the non-vanishing section. Hence $c_1(X) = 0$.*

Definition 2: *A complex, compact n-dimensional Kähler manifold X is Calabi-Yau if $c_1(X) = 0$ and $h^{0,1}(X) = h^{0,2}(X) = \cdots = h^{0,n-1}(X) = 0$*

Remark: *$c_1(X) = 0$ implies Ricci curvature $= 0$ which is Einstein's field equations of GTR in vacuum.*

Remark: Calabi-Yau manifolds come in pairs (X, X^*) where X^* is called Mirror manifold such that $h^{1,1}(X) = h^{1,2}(X^*)$ and $h^{1,2}(X) = h^{1,1}(X^*)$

Remark: $\chi(X) = 2(h^{1,1} - h^{2,1})$ is the Euler Characteristic of X and $\chi(X^*) = -\chi(X)$ where X^* is the mirror CY.

Cohomological Properties of Hodge Parameters $h^{p,q}(X)$

1. $h^{p,q}(X) = h^{q,p}(X)$ (Dolbeault Property) 2. $h^{p,q}(X) = h^{n-p,n-q}(X)$ (Serre Duality)

THe holomorphic Euler-Poincaré characteristic $\chi^p(X) = \sum_{q=0}^n(-1)^q h^{p,q}(X)$ The generating function of $\chi^p(X)$ is denoted by: $\chi_y(X) = \chi(y, X) = \sum_{p\geq 0}\chi^p(X)y^p$. From Hirzebruch, the $\chi_y(X)$ is the coefficient of ξ^N in the ξ Taylor expansion of the function: $f(N,y,\xi) = \frac{(1+(1-\xi))^{N+!}}{(1-\xi)(1+y)} \cdot \frac{1-(1-\xi)^m}{1+y(1-\xi)^m}$. $f(N,y,\xi) = \sum_{p\geq 0} f(N,p,\xi)y^p$. $\chi^0(X) = f(N,0,\xi) = \frac{1}{1-\xi} - (1-\xi)^{m-1}$ and $\chi^1(X) = f(N,1,\xi) = (N+1)(1-(1-\xi)^m) - \frac{1}{1-\xi} + (1-\xi)^{2m-1}$

$c_1(X) = 0$ for a hyper-surface $X \subset \mathbb{CP}^n$ can be reformulated as $d = n+1$ where d is the degree of the homogeneous polynomial $F(z_0, \ldots, z_n)$.

$X \hookrightarrow \mathbb{CP}^{n_1}_1 \times \cdots \mathbb{CP}^{n_m}_m$ then $f_1, ..., f_k$ are homogeneous polynomials restricted to $\mathbb{CP}^{n_r}_r$ of degrees $d^r_1 \ldots d^r_k$ then $c_1(X) = 0$ of the complete intersection CY is given by $\sum_{a=1}^k d^r_a = n_r + 1$ for $r = 1, \ldots, m$.

Calabi-Yau Quintic Threefold

$$X = \{(z_0 : z_1 : z_2 : z_3 : z_4) \in \mathbb{CP}^4 : z_0^5 + z_1^5 + z_2^5 + z_3^5 + z_4^5 - 5\psi z_0z_1z_2z_3z_4 = 0\}$$

Tian-Yau CICY Threefold

$$X = \{(x_0 : x_1 : x_2 : x_3 : y_0 : y_1 : y_2 : y_3) \in \mathbb{CP}^3 \times \mathbb{CP}^3 : x_0^3 + x_1^3 + x_2^3 + x_3^3 = 0, y_0^3 + y_1^3 + y_2^3 + y_3^3 = 0, x_0y_0 + x_1y_1 + x_2y_2 + x_3y_3 = 0\}$$

Cohomological Parameters $h^{0,0}, h^{3,0}, h^{1,1}, h^{1,2}$

For the **CY quintic threefold**, $\chi^0(X) = 1 - \frac{(m-1)(m-2)(m-3)(m-4)}{24}$ then $h^{0,0}(X) = 1$ and $h^{0,3}(X) = \frac{(m-1)(m-2)(m-3)(m-4)}{24}$. $\chi^1(X) = -5(\frac{m(m-1)(m-2)(m-3)}{24}) - 1 + (\frac{(2m-1)(2m-2)(2m-3)(2m-4)}{24}$. For $m = 5$, then $\chi^1(X) = -1 + 101 = -h^{1,1} + h^{1,2}$. Hence, $h^{0,0}(X) = 1, h^{3,0}(X) = 1, h^{1,1}(X) = 1, h^{2,1}(X) = 101$.

$G_X = \langle g_1, g_2\rangle$ where $g_1 \cdot (x_0, x_1, x_2, x_3, x_4) = (x_0, \xi x_1, x_2, x_3, \xi^4 x_4)$ and $g_2 \cdot (x_0, x_1, x_2, x_3, x_4) = (x_0, x_1, \xi^2 x_2, \xi^3 x_3, x_4)$, where $\xi^5 = 1$ Then, $G_X \times X \to X$ is free hence number of generations of fermions is $|\chi(Y)|/2 = \frac{|\chi(X)|}{2|G_X|} = 4$, where $Y = X/G_X$.

For the CICY threefold, $\chi(X) = [\sum_{r,s,t=1}^2(\frac{1}{3}[\delta^{r,s,t}(n_r + 1) - \sum_{a=1}^3 d^r_a d^s_a d^t_a])x_r x_s x_t \cdot \bigwedge_{b=1}^3(\sum_{p=1}^2 d^p_b x_p)]_{\text{top}} = -18$ where $n_1 = n_2 = 3$, $d^1_1 = d^1_2 = 1, d^1_2 = d^2_3 = 1, d^1_3 = d^2_2 = 0$. where there are 40 free parameters for degree 3 polynomials in 4 variables. Then, 16 from linear automorphism which does not change the shape of X and 1 is the overall scaling, hence $h^{1,2} = 40 - 16 - 1 = 23$ and $h^{1,1}(X) = 14$ from Euler Characteristic. $G_X \cong \mathbb{Z}_3$ has a free action on X, hence number of generations of fermions is $|\chi(Y)|/2 = \frac{|\chi(X)|}{2|G_X|} = 3$, where $Y = X/G_X$.

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