Correspondence Between the Witten-Atiyah-Segal TQFT Axioms of the Symmetric Modular Functor and the Eilenberg-Steenrod Axioms of the Homology Functor .

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Introduction [Math Motivation]

In 1988 Michael Atiyah published his "Topological Quantum Field Theory" paper in which he formulated TQFT into an axiomatic system. This formulation uses category theory as it describes TQFT as a *functor* between the cobordism category and the category of R-modules. In this project, our goal is to explore the correspondence between the axioms of the symmetric modular functor (constructed by Graeme Segal in the context of conformal field theory in 1989) with another well-known axiomatic system related to homology theory developed in 1952 by mathematicians Samuel Eilenberg and Norman Steenrod in "Foundations of Algebraic Topology" known as the homology functor.

A *category* C consists of a collection $Obj(C)$ of objects and a collection $Mor(C)$ of morphisms such that morphism composition is associative and there exists identity morphisms. A *functor* F between two categories C &C' pairs each $X \in Obj(C)$ with an object $F(X) \in Obj(C')$ and each $f \in Mor(C)$ with a morphism $F(f) = f' \in Mor(C')$. It was noted by Atiyah that the axioms of TQFT have marked similarity with the axioms of homology theory in Algebraic Topology. We briefly describe these functors. For each $q \in \mathbb{Z}^+$ the homology functor is denoted by $H_q: Top^{(2)} \to AbGrp$ and the TQFT functor is denoted by $\tau_q: qCob \to Vect_K$ where:

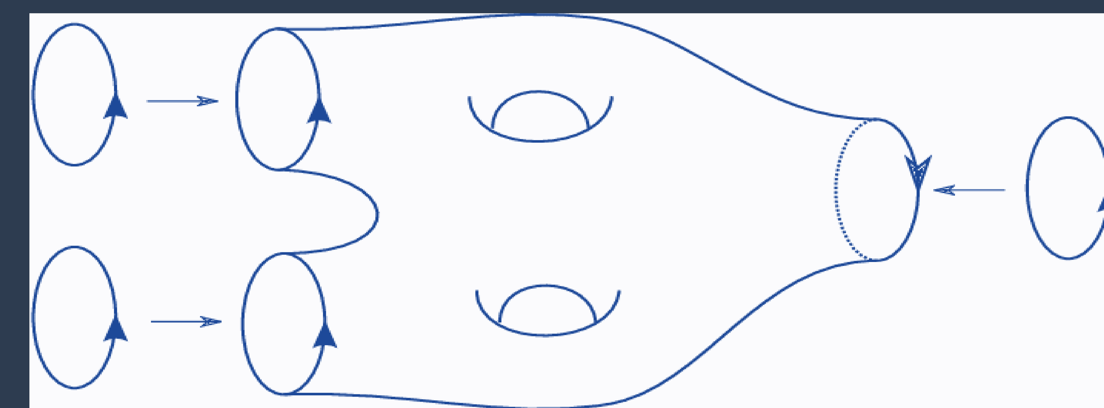
- $Top^{(2)}$ is the category of pairs of topological spaces with morphisms that are continuous functions.
- $AbGrp$ is the category of abelian groups with morphisms that are homomorphisms.
- $Vect_K$ is the category of vector spaces over a field K with morphisms that are isomorphisms.
- $qCob$ is the category of q-dimensional compact oriented topological manifolds with morphisms that are classes of *cobordisms* which are (q+1) dimensional oriented topological manifolds with boundaries homeomorphic to the objects.
  - Remark: $qCob$ and $Vect_K$ may be thought of as *symmetric monoidal* categories. If C is a category with an operation ■ and objects and C then the collection $(C, \blacksquare, \eta, \alpha, \lambda, \rho, \tau, I)$ is a symmetric monoidal category with the properties:
    - $\eta: 1 \to C$ with $I \in 1$ is the neutral element and the only object in 1.
    - $((A \blacksquare B) \blacksquare C) \cong (A \blacksquare (B \blacksquare C))$ given by the associator object α.
    - $(I \blacksquare A) \cong A$ & $(A \blacksquare I) \cong A$ given by λ & ρ respectively.
    - $A \blacksquare B \cong B \blacksquare A$ given by τ.

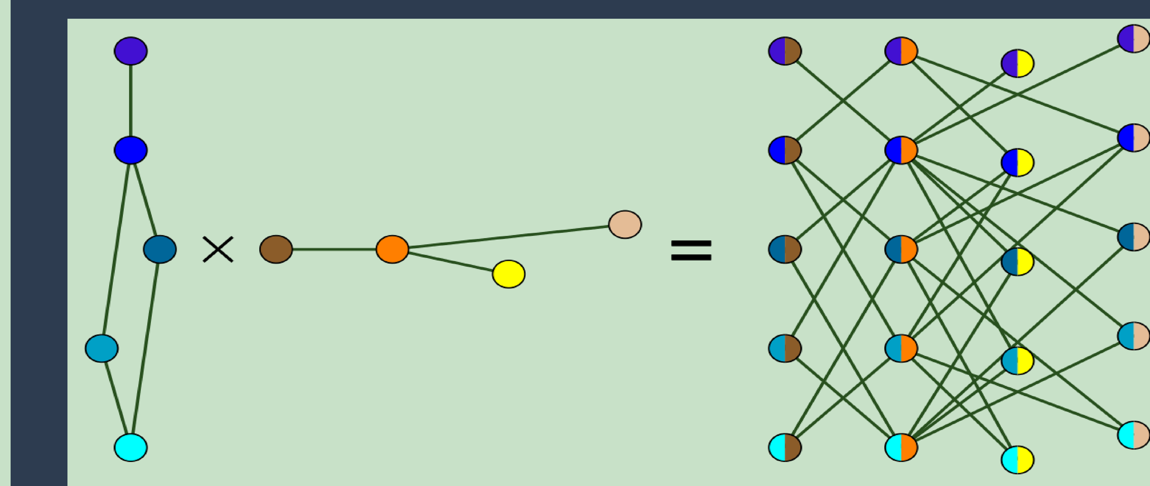
The Eilenberg-Steenrod axioms of Homology theory are satisfied by the qth homology functor H_q which associates each pair of topological spaces $(X, A) \in Obj(Top^{(2)})$ with an abelian group $H_q((X, A)) \in Obj(AbGrp)$ and each continuous function $f \in Mor(Top^{(2)})$ with a homomorphism $\mathrm{H_q(f)} = f_* \in Mor(AbGrp)$.

The Witten-Atiyah-Segal axioms of TQFT are satisfied by the *symmetrical modular* functor τ_q which associates each topological manifold $\Sigma \in Obj(qCob)$ with a vector space $\tau_q(\Sigma) \in Obj(Vect_k)$ and each cobordism $M \in Mor(qCob)$ with a vector $\tau_q(M) \in \tau_q(\partial M) \in Obj(Vect_K)$.

Notations:

- $\sqcup$ denotes a coproduct
- $\oplus$ denotes a direct product
- $\otimes$ denotes a tensor product





Methods

The goals of this project were pursued by means of detailed analysis of the works of Ed Witten, Michael Atiyah and Graeme Segal in the fields of Topological Quantum Field Theory and Conformal Field Theory, as well as the works of Samuel Eilenberg and Norman Steenrod in the field of Algebraic Topology. Additionally, we referred to various works from mathematicians, as listed in the references, in order to correctly construct a formulation for topics such as the qCob category, the definition of a monoidal category, the definition of a symmetric tensor category and more. Over the course of the program, we constructed an informal paper which establishes a detailed formulation for the theories needed to understand the axiomatic systems at hand. When the axioms were established, we used them to see what theorems we could immediately derive, as well as direct comparison of the axioms and their implications.

Results

<table>
<tr><th>E-S Axioms satisfied by H_q</th><th>W-A-S Axioms satisfied by τ_q</th></tr>
<tr><td>(i) If $id : (X, A) \to (X, A)$, then $H_q(id) = id_* : H_q(X, A) \to H_q(X, A)$.</td><td>(i) If $Cyl = \Sigma \times [0,1]$ then $\partial(Cyl) = \Sigma \sqcup \bar{\Sigma}$ and $\tau_q(\Sigma \times [0,1]) = id_* : \tau_q(\Sigma) \to \tau_q(\Sigma)$</td></tr>
<tr><td>(ii) If $f : (X, A) \to (Y, B)$ and $g : (Y, B) \to (Z, C)$ with $gf : (X, A) \to (Z, C)$ then $H_q(gf) = H_q(g)H_q(f) = (gf)_* : H_q(X, A) \to H_q(Z, C)$.</td><td>(ii) [Unitary Axiom]: If $M : \Sigma_0 \Longrightarrow \Sigma_1$ is a cobordism and if $f : \Sigma_0 \to \Sigma_1$ is a homeomorphism, then $\tau_q(f) = f_* : \tau_q(\Sigma_0) \to \tau_q(\Sigma_1)$ is an isomorphism and $\tau_q(\bar{\Sigma}) = \tau_q(\Sigma)^*$</td></tr>
<tr><td>(iii) The following diagram commutes:

$$\begin{array}{ccc} H_q(X,A) & \xrightarrow{f_*} & H_q(Y,B) \\ \downarrow \partial & & \downarrow \partial \\ H_{q-1}(A) & \xrightarrow{f|_A} & H_{q-1}(B) \end{array}$$

In other words $\partial f_* = (f|_A)_* \partial$</td><td>(iii) Let $M : \Sigma_0 \Longrightarrow \Sigma_1$ and $M' : \Sigma_0' \Longrightarrow \Sigma_1'$ where $\partial M = \Sigma_0 \sqcup \bar{\Sigma}_1$ and $\partial M' = \Sigma_0' \sqcup \bar{\Sigma}_1'$. If $f : M \to M'$ is a homeomorpshism with $i = i_0 \sqcup i_1 : \Sigma_0 \sqcup \Sigma_1 \to M$ and $i' = i_0' \sqcup i_1' : \Sigma_0' \sqcup \Sigma_1' \to M'$ are continuous inclusion maps then from the following diagram:

$$\begin{array}{ccc} M & \xrightarrow{f} & M' \\ \uparrow i & & \uparrow i' \\ \partial M & \xrightarrow{f|_{\partial M}} & \partial M' \end{array}$$

we have the isomorphism $(f|_{\partial M})_* : \tau_q(\partial M) \to \tau_q(\partial M')$ i.e. $(f|_{\partial M})_* \tau_q(\partial M) = \tau_q(\partial M')$.</td></tr>
<tr><td>(iv) If i_* and j_* represent inclusion maps, then the following sequence is *exact*:

$$\ldots \to^{\partial_{q+1}} H_q(A) \to^{i_{*q}} H_q(X) \to^{j_{*q}} H_q(X, A) \to^{\partial_q} H_{q-1}(A) \to^{i_{*q-1}} \ldots$$

Exactness $\Rightarrow$ *image* = *kernel*. Example: $\partial_q \circ j_* = 0$.</td><td>(iv) [Locality Axiom]: A cobordism (M, Σ_0, Σ_2) may be obtained by the *gluing* of two cobordisms (M', Σ_0, Σ_1) and $(M'', \Sigma_1', \Sigma_2)$ along a homeomorphism $h : \Sigma_1 \to \Sigma_1'$ denoted by $M \cong M' \sqcup M''/h \ni \partial M \cong \partial M' \sqcup \partial M''$. Similarly, if M is a coborism with boundary ∂M and if $\Sigma \hookrightarrow M$ is a closed, oriented, codimension-one topological sub-manifold, then a cobordism $\breve{M}$ may be obtained by *cutting* along $\Sigma \ni \partial\breve{M} \cong \partial M \sqcup \Sigma \sqcup \bar{\Sigma}$.</td></tr>
<tr><td>(v) [Homotopy Axiom] If $f : (X, A) \to (Y, B)$ and $g : (X, A) \to (Y, B)$ such that f and g are *homotopic*, then $f_* = g_* : H_q(X, A) \to (Y, B)$ is unique.</td><td>(v) [Isotopy Axiom]: If $f, g : \Sigma_0 \to \Sigma_1$ are *isotopic* (or in the same isotopy class) then $f_* = g_* : \tau_q(\Sigma_0) \to \tau_q(\Sigma_1)$ is unique.</td></tr>
<tr><td>(vi) [Additive Axiom]: If $(X_1, A_1), (X_2, A_2) \in Obj(Top^{(2)})$ then:

$$H_q((X_1, A_1) \sqcup (X_2, A_2)) \cong H_q(X_1, A_1) \oplus H_q(X_2, A_2)$$</td><td>(vi) [Multiplicative Axiom]: If $\Sigma_0, \Sigma_1 \in Obj(qCob)$, then

$$\tau_q(\Sigma_0 \sqcup \Sigma_1) = \tau_q(\Sigma_0) \otimes_K \tau_q(\Sigma_1)$$</td></tr>
<tr><td>(vii) If $pt \in Obj(Top^{(2)})$ is a single point space, then $H_q(pt) = 0$ $(\forall q \in \mathbb{Z}^+)$.</td><td>(vii) If ϕ_q is the *empty* cobordism, then $\tau_q(\phi_q) = K$ $(\forall q \in \mathbb{Z})$</td></tr>
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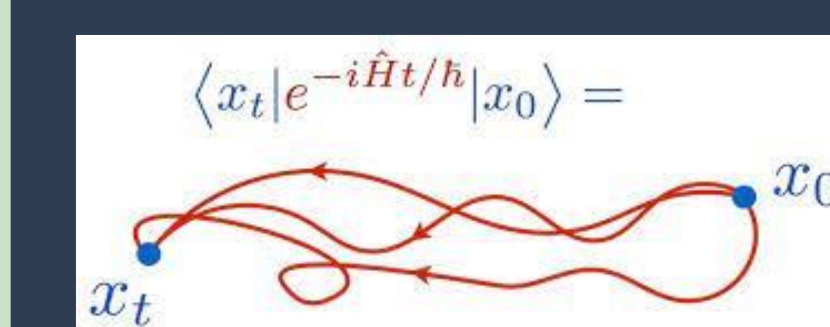

Physics Motivation

Suppose we have some path $\gamma: [t_0, t_1] \to M$ that lies on a cobordism [spacetime] M such that $\gamma(t_0) = a$ & $\gamma(t_1) = b$ with $a, b \in M$. We say $\gamma \in \wp$ where $\wp$ is the *Space of paths.* The action integral for γ is given by $s[\gamma] = \int_{t_0}^{t_1} L(t, \gamma(t), \gamma'(t))dt$. In quantum mechanics we may compute the contribution of any path $\gamma \in \wp$ by integrating the probability function over the path γ with respect to the space of paths $\wp$ i.e., $\int_{\gamma \in \wp} e^{\frac{i\,s[\gamma]}{\hbar}} d\wp$. This is the famous Feynman path integral. Now, we want to compute the probability amplitude for a given field configuration to evolve from a space Σ_0 to a space Σ_1. Suppose M is a cobordism, denoted by $M: \Sigma_0 \Longrightarrow \Sigma_1$, with boundary $\partial M = \Sigma_0 \sqcup \overline{\Sigma_1}$. Let $\mathcal{A}(\Sigma_0)$ represent the initial fields "living" on a space Σ_0. That is to say, $\mathcal{A}(\Sigma_0)$ is the field configuration on Σ_0. The Schrödinger interpretation states that the probability of the state ψ_t being observed in a field configuration $\mathrm{A} \in \mathcal{A}(\Sigma_t)$ is represented as $\psi_t(A) = \hat{A}_t \in Vect_K$ (usually some Hilbert space). The probability amplitude for a given field configuration $A_0 \in \mathcal{A}(\Sigma_0)$ to evolve into $A_1 \in \mathcal{A}(\Sigma_1)$ is given by $\langle \hat{A}_1 | U_t | \hat{A}_0 \rangle = \int_{A_0}^{A_1} e^{\frac{i\,s[A]}{\hbar}} d\mathcal{A}(\Sigma)$ Hence, the Feynman path integral constructs the time evolution operator U_t associated with the cobordism M. Hence, for each $A_t \in \mathcal{A}(\Sigma_t)$ we associate the space of states $[\hat{A}_t \in \tau_q(\Sigma_t) \in Vect_K]$ and for each $M_t: \Sigma_0 \Longrightarrow \Sigma_t$ we associate $U_t(M) = \tau_q(M_t) \in \tau_q(\partial M_t)$. Therefore, QFT may be generalized as computing $\tau_q(M_t) = U_t(M)$ where $\tau_q(M)$ is a topological invariant for the manifold M. However, this process often leads to an ill-defined integral. There are special cases that may be solved, and the three-dimensional case was computed and interpreted by Ed Witten as the partition function $\tau_{q,k}(M) = Z_k(M) = \int_A e^{\frac{i\,k\,s[A]}{\hbar}} d\mathcal{A}$. This is known as one of the defining moments of TQFT because this partition function is independent of a metric. If g is some metric, then $\frac{\delta}{\delta g}(Z_k) = 0$.

Conclusion

We found that the two axiomatic systems correlate in some ways while also representing different mathematical objects and leading to different outcomes. The role of an axiomatic system is to provide a means for proving results in the field in the form of theorems. Simply due to the difference of categories, there must be some axioms which do not correlate. The obvious case would be the fourth axioms. The concept of an exact sequence is very homological in nature and the gluing\cutting of cobordisms is of course unique to cobordisms. Thus, the fourth axioms do not compare. The following are a few interesting results that we noticed of the W-A-S axioms:

- From the second W-A-S axiom, we may derive a formulation that exemplifies the covariant nature of the τ_q functor. That is $f: \Sigma_0 \to \Sigma_1, g: \Sigma_1 \to \Sigma_2 \in Mor(qCob) \Rightarrow (f \circ g)_* = f_* \circ g_*: \tau_q(\Sigma_0) \to \tau_q(\Sigma_2)$. Notice the similarity between this formulation and E-S (ii).
- The following remark is a result of W-A-S (iv). $\tau_q(M) \in \tau_q(\Sigma_0 \sqcup \bar{\Sigma}_2) = \tau_q(\Sigma_0) \otimes \tau_q(\bar{\Sigma}_2) = \tau_q(\Sigma_0) \otimes \tau_q(\Sigma_2)^*$. In the case where M is obtained by the gluing of M'& M'' along the homeomorphism $h: \Sigma_1 \to \overline{\Sigma_1}$, we have $\tau_q(M) \in \tau_q(\partial(M' \sqcup M'')) = \tau_q(\partial M' \sqcup \partial M'') = \tau_q(\partial M') \otimes \tau_q(\partial M'') = \tau_q(\Sigma_0) \otimes \tau_q(\Sigma_1)^* \otimes \tau_q(\Sigma_1) \otimes \tau_q(\Sigma_2)^* \to \tau_q(\Sigma_0) \otimes \tau_q(\Sigma_2)^*$.
- The third W-A-S axiom is very similar to E-S (iii). Consider the following: if $\partial: \Sigma_0 \sqcup \Sigma_1 \rightrightarrows \partial M_{in} \sqcup \partial M_{out} \to^{i} \partial M$ and $\partial': \Sigma'_0 \sqcup \Sigma'_1 \rightrightarrows \partial M'_{in} \sqcup \partial M'_{out} \to^{i'} \partial M' \Rightarrow (fi\partial)_*(\tau_q(\Sigma_0 \sqcup \Sigma_1)) = (i'\partial\, f|_{\partial M})_*(\tau_q(\Sigma_0 \sqcup \Sigma_1)) \Rightarrow f\partial_* = \partial_*(f|_{\partial M})_*$.
- It must be noted that the axiom W-A-S (vi) is what causes this system to be quantum in nature. The vector spaces associated with two topological manifolds may be combined as a tensor product just as two quantum states may be superimposed.
- We have the following result from W-A-S (vii). If M is a (2+1) cobordism without boundary, then $\tau_q(M) \in \tau_q(\partial M) = \tau_q(\phi) = K$ which implies that $\tau_q(M) \in K$. $\tau_q(M)$ is a *numerical invariant* for M computed and interpreted by Ed Witten as $\tau_{q,k} = \int_A e^{\frac{i\,k\,cs[A]}{\hbar}} d\mathcal{A}$ where $cs[A] = \frac{1}{4\pi}\int_M Tr\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right) d(Vol)$ where$A \in \mathcal{A}$ is the field configuration or space of fields.

References

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