The Proportion of Comets in the Card Game SET

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Fact: If A and B are distinct, then A, B, and $(-A - B)$ are all distinct.

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$$\sum_{x \in S} x = 0$$

Some code

```
import itertools

all_cards = [p for p in itertools.product(range(3),
                                          repeat = 4)]

for k in xrange(10):
    print sum(all(sum(x[i] for x in p)%3 == 0
        for i in xrange(4))
        for p in itertools.combinations(all_cards, k))
```

OK, this is taking a while. . .

k	subsets of size k that sum to 0	running time (seconds)
0	1	0.01
1	1	0.01
2	40	0.02
3	1080	0.23
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(I gave up before I got to $k = 9$.)

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Suffices to count the 8-card subsets which already do contain the "required" card:

$$\left(-\sum_{x \in S} x\right) \in S$$

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(Some bookkeeping remains: have to deal with over-counting, *etc.*)

The Recurrence Relations

Define

$$A_k = \left\{ S \subseteq D \middle| \left(\sum_{x \in S} x \right) \in S, \text{ and } |S| = k \right\},$$

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and let $a_k = |A_k|$, and $b_k = |B_k|$, and $c_k = |C_k|$.

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(We originally wanted c_9.)

The Recurrence Relations

Theorem

$$a_{k+1} = (c_k)(81 - k)$$

$$b_{k+1} = \binom{81}{k} - a_k$$

$$c_{k+1} = \frac{\binom{81}{k} - b_k}{k+1}$$

□

Results

k	a_k	b_k	c_k
0	0	0	1
1	81	1	1
2	80	0	40
3	3160	3160	1080
4	84240	82160	20540
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Let's write ${}_na_k = |{}_nA_k|$.

Theorem

For $k > 0$:

$$ {}_{\infty}a_k = \frac{\binom{|V|}{k-1} - {}_{(p-1)}a_{k-1}}{k}, $$

$$ {}_{1}a_k = {}_{\infty}a_{k-1}\left(|V| - (k-1)\right), $$

and for $n \notin \{1, \infty\}$,

$$ {}_{n}a_k = \binom{|V|}{k-1} - {}_{\left(\frac{n}{1-n}\right)}a_{k-1}. $$

Further, we have the initial values

$$ {}_{n}a_0 = \begin{cases} 0, & \textit{if } n \neq \infty \\ 1, & \textit{if } n = \infty \end{cases} $$

$\square$

Corollary

For $k > 0$ and $n \in \{1, \ldots, p-1, \infty\}$, we can write ${}_n a_k$ in terms of ${}_{(\frac{n}{1-n})} a_{k-1}$, where by $\frac{n}{1-n}$ we mean $n \cdot (1-n)^{-1} \pmod{p}$, and in particular we define $\frac{\infty}{1-\infty} = -1 = p-1$, and $\frac{1}{1-1} = \infty$. $\square$

This $\sigma(n) = \frac{n}{1-n}$ is a permutation on $\{1, \ldots, p-1, \infty\}$, and its order is p (it is a p-cycle.)

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To see this, notice that σ is a fractional linear transformation; it can be written as

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To calculate ${}_n a_k$, visit all the other ${}_m a_*$ sequences, and return to ${}_n a_*$ after exactly p steps; *i.e.*, at ${}_n a_{k-p}$.

Example

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Next we calculate ${}_3 a_2$, and ${}_6 a_3$, and ${}_\infty a_4$, and keep going until we get to ${}_\infty a_{18}$.

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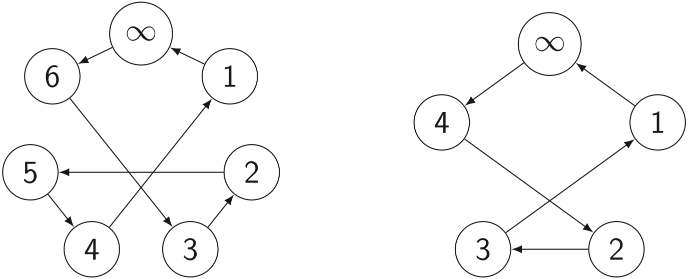
This procedure allows us to calculate only the coefficients that we need.

Thank you!



(The permutation σ in characteristic 7 and characteristic 5)

References

[1] *Sets, Planets, and Comets*. Mark Baker, Jane Beltran, Jason Buell, Brian Conrey, Tom Davis, Brianna Donaldson, Jeanne Detorre-Ozeki, Leila Dibble, Tom Freeman, Robert Hammie, Julie Montgomery, Avery Pickford, and Justine Wong. *The College Mathematics Journal*, **Vol. 44**, No. 4 (September 2013), pp. 258-264